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# ALGORITHM FOR GENERALIZED MULTIVALUED VARIATIONAL INEQUALITIES IN HILBERT SPACES

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**Abstract**—We introduce a new class of generalized multivalued variational inequality problems (1.1) which contain several existing variational inequality problems as special cases. An iterative algorithm for finding approximate solutions of problem (1.1) is considered. Several convergence results for this algorithm are derived and in particular several existence results of problems (1.1) are obtained. We also introduce a new class of generalized multivalued complementarity problems (5.1) which also contain several known complementarity problems as special cases. It is shown that problem (5.1) is equivalent to problem (1.1) of special type from which several existence results of problem (5.1) are obtained.

## 1. INTRODUCTION

Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$, respectively. Let $T, A, G : H \to H$, $V : H \to C(H)$, where $C(H)$ is the family of nonempty compact subsets of $H$, $K : H \to 2^H$ such that $K(x)$ is nonempty closed convex for all $x \in H$. We shall study in this paper the following generalized multivalued variational inequality: find $x \in H$, $y \in V(x)$ such that $Gx \in K(x)$ and

$$(Tx + Ay, z - Gx) \geq 0, \qquad \text{for all } z \in K(x). \tag{1.1}$$

Before we proceed any further, we make the following observations.

(i) If $T \equiv 0$, $G$ is the identity mapping on $H$, $K(x) = m(x) + X$, where $m$ is a point-to-point mapping, $X$ is a closed convex subset of $H$, then inequality (1.1) is equivalent to finding $x \in K(x)$, $y \in V(x)$ such that

$$(Ay, z - x) \geq 0, \qquad \text{for all } z \in K(x). \tag{1.2}$$

Inequalities like (1.2) are known as generalized multivalued quasivariational inequalities introduced and studied by Chang and Huang [1].

(ii) If $V$ is the identity mapping on $H$ and $A$ is replaced by $-A$, then inequality (1.1) is equivalent to finding $x \in H$ such that $Gx \in K(x)$ and

$$(Tx - Ax, z - Gx) \geq 0, \qquad \text{for all } z \in K(x). \tag{1.3}$$

Inequalities like (1.3) are known as generalized strongly nonlinear variational inequalities studied independently by Siddiqi and Ansari [2] and Guo and Yao [3].

(iii) If $A$ is replaced by $-A$, $G$ and $V$ are the identity mappings on $H$, then inquality (1.1) is equivalent to finding $x \in H$ such that $x \in K(x)$ and

$$(Tx - Ax, z - x) \geq 0, \qquad \text{for all } z \in K(x). \tag{1.4}$$

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Inequalities like (1.4) are known as strongly nonlinear quasivariational inequalities studied by Siddiqi and Ansari [4].

(iv) If $A \equiv 0$, $V$ is the identity mapping on $H$, then inequality (1.1) reduces to the problem of finding $x \in H$ such that $Gx \in K(x)$ and

$$(Tx, z - Gx) \geq 0, \qquad \text{for all } z \in K(x). \tag{1.5}$$

The problem (1.5) is the quasivariational inequality problem studied by Noor [5].

(v) If $K(x) = X$ for all $x \in H$ where $X$ is a nonempty closed convex subset of $H$, $V$ and $G$ are the identity mappings on $H$ and $A$ is replaced by $-A$, then inequality (1.1) is equivlent to finding $x \in X$ such that

$$(Tx - Ax, z - x) \geq 0, \qquad \text{for all } z \in X. \tag{1.6}$$

Inequalities like (1.6) are known as the mildly nonlinear variational inequalities which were introdution by Noor [6].

(vi) If $K(x) = X$ for all $x \in H$ where $X$ is a nonempty closed convex subset of $H$, $A \equiv 0$, $G$ and $V$ are identity mappings on $H$, then inequality (1.1) is equivalent to finding $x \in X$ such that

$$(Tx, z - x) \geq 0, \qquad \text{for all } z \in X. \tag{1.7}$$

Inequalities like (1.7) are known as the classical variational inequalities which have been extensively studied in the literature both in finite and infinite dimensional spaces [7–9].

Therefore, problems (1.2)–(1.7) are special cases of the problem (1.1). In summary, we conclude that the problem (1.1) is a more general and unifying one, which is the main motivation of this paper. In Section 2, we shall give some preliminaries that will be used throughout this paper. In Section 3, we employ the projection method to formulate characterization of solutions of problem (1.1). In Section 4, we contruct an algorithm for finding the approximate solutions of problem (1.1), and derive some existence and convergence results. In the final section, we consider a generalized multivalued complementarity problem (5.1). We show that problem (5.1) is equivalent to problem (1.1) of special type and from which we obtain some existence and convergence results for problem (5.1).

## 2. PRELIMINARIES

We first recall the following definitions.

DEFINITION 2.1. *The mapping $A : H \to H$ is said to be*

(i) *strongly monotone with respect to the point-to-set mapping $V : H \to C(H)$ if there exists a constant $\alpha > 0$ such that*

$$(Au - Av, x - y) \geq \alpha \|x - y\|^2, \qquad \text{for all } x, y \in H \text{ and for all } u \in V(x),\ v \in V(y).$$

(ii) *Lipschitz continuous if there exists a constant $\beta > 0$ such that*

$$\|Ax - Ay\| \leq \beta \|x - y\|, \qquad \text{for all } x, y \in H.$$

DEFINITION 2.2. *The mapping $T : H \to H$ is said to be strongly monotone if there exists a constant $\gamma > 0$ such that*

$$(Tx - Ty, x - y) \geq \gamma \|x - y\|^2, \qquad \text{for all } x, y \in H.$$

We note that if the mapping $T$ is both strongly monotone with $\gamma > 0$ and Lipschitz continuous with constant $\beta > 0$, then $\gamma \leq \beta$.

DEFINITION 2.3. *The mapping $V : H \to C(H)$ is said to be* **H**-*Lipschitz continuous if there exists a constant $\eta > 0$ such that*

$$\mathbf{H}(V(x), V(y)) \leq \eta \|x - y\|, \qquad \text{for all } x, y \in H,$$

*where $\mathbf{H}(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$.*

For any closed nonempty convex subset $X$ of $H$, let $P_X(\cdot)$ be the projection mapping of $H$ onto $X$.

LEMMA 2.4. [7, Theorem 2.3, pp. 9] *Given $z \in H$, $x = P_X(z)$ if and only if $x \in X$ and*

$$(x - z, y - x) \geq 0, \qquad \text{for all } y \in X.$$

LEMMA 2.5. [7, Corollary 2.4, pp. 10] *The mapping $P_X(\cdot)$ is nonexpansive, i.e.,*

$$\|P_X(x) - P_X(y)\| \leq \|x - y\|, \qquad \text{for all } x, y \in H.$$

LEMMA 2.6. [10,11] *Let $K(x) = m(x) + X$ for all $x \in H$. Then*

$$P_{K(x)}(y) = m(x) + P_X(y - m(x)).$$

For any nonempty subset $Y$ of $H$, we recall that the *dual set* of $Y$ is

$$Y^* = \{x \in H : (x, y) \geq 0, \qquad \text{for all } y \in Y\}.$$

LEMMA 2.7. [12] *If $K(x) = m(x) + X$ for all $x \in H$, then*

$$K^*(x) = m^*(x) \cap X^*.$$

## 3. CHARACTERIZATION OF SOLUTIONS

By Lemma 2.4, we have the following characterization of solutions of the problem (1.1).

THEOREM 3.1. *Let $H$ be a real Hilbert space, $T, A, G : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x)$ is nonempty closed covex for all $x \in H$. The following statements are equivalent:*

(i) *$x \in H$, $y \in V(x)$ are solutions of (1.1).*
(ii) *$x \in H$, $y \in V(x)$ and $Gx = P_{K(x)}(Gx - \rho(Tx + Ay))$ for some $\rho > 0$.*

PROOF. (i) implies (ii). Since $x \in H$, $y \in V(x)$ are solutions of (1.1), $Gx \in K(x)$ and

$$(Tx + Ay, z - Gx) \geq 0, \qquad \text{for all } z \in K(x).$$

By Lemma 2.4, we then have

$$Gx = P_{K(x)}(Gx - (Tx + Ay)),$$

and hence (ii) holds.

(ii) implies (i). Since $x \in H$, $y \in V(x)$ and

$$Gx = P_{K(x)}(Gx - \rho(Tx + Ay)), \qquad \text{for some } \rho > 0,$$

$Gx \in K(x)$. By Lemma 2.4 again, we have

$$(\rho(Tx + Ay), z - Gx) \geq 0, \qquad \text{for all } z \in K(x).$$

By dividing $\rho$ on both sides of the above inequality, we have

$$(Tx + Ay, z - Gx) \geq 0, \qquad \text{for all } z \in K(x),$$

and hence (i) holds.

From Theorem 3.1, we have the following equivalent result.

THEOREM 3.2. *Let $H$ be a real Hilbert space, $T, A, G : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x)$ is nonempty closed convex for all $x \in H$. Then $x \in H$, $y \in V(x)$ are solutions of (1) if and only if*

$$x = x - Gx + P_{K(x)}(Gx - \rho(Tx + Ay)), \qquad \text{for some } \rho > 0.$$

By Lemma 2.6 and Theorem 3.2, we have the following theorem.

THEOREM 3.3. *Let $H$ be a real Hilbert space, $X$ a nonempty closed convex subset of $H$, $T, A, G, m : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x) = m(x) + X$ for all $x \in H$. Then $x \in H$, $y \in V(x)$ are solutions of (1.1) if and only if*

$$x = x - Gx + m(x) + P_X(Gx - \rho(Tx + Ay) - m(x))$$

*for some $\rho > 0$.*

## 4. ALGORITHM, EXISTENCE AND CONVERGENCE

We now construct an algorithm for finding approximate solutions of the problem (1.1). Let $K(x) = m(x) + X$ where $X$ is a nonempty closed subset of $H$ and $\rho > 0$ be fixed.

Given $x_0 \in H$, take any $y_0 \in V(x_0)$ and let

$$x_1 = x_0 - Gx_0 + m(x_0) + P_X(Gx_0 - \rho(Tx_0 + Ay_0) - m(x_0)).$$

Since $V(x_0)$ is a nonempty and compact set, there exists $y_1 \in V(x_1)$ such that

$$\|y_0 - y_1\| \leq \mathbf{H}(V(x_0), V(x_1)).$$

Let

$$x_2 = x_1 - Gx_1 + m(x_1) + P_X(Gx_1 - \rho(Tx_1 + Ay_1) - m(x_1)).$$

Continuing the above process inductively, we can get sequences $\{x_n\}$ and $\{y_n\}$ such that

$$y_n \in V(x_n), \|y_n - y_{n+1}\| \leq \mathbf{H}(V(x_n), V(x_{n+1})) \tag{4.1}$$

and

$$x_{n+1} = x_n - Gx_n + m(x_n) + P_X(Gx_n - \rho(Tx_n + A_n) - m(x_n)), \qquad n = 1, 2, 3, \ldots \tag{4.2}$$

We have the following existence and convergence results.

THEOREM 4.1. *Let $H$ be a real Hilbert space, $X$ a nonempty closed convex subset of $H$, $T, A, G, m : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ and $K(x) = m(x) + X$ for all $x \in H$. Suppose that the following conditions are satsfied:*

(i) *$T$ is Lipschitz continuous with constant $\beta$,*
(ii) *$G$ is both strongly monotone with constant $\gamma > 0$ and Lipschitz continuous with constant $\delta$,*
(iii) *$A$ is both strongly monotone with respect to $V$ with constant $\kappa > 0$ and Lipschitz continuous with constant $\lambda$,*
(iv) *$m$ is Lipschitz continuous with constant $\theta$,*
(v) *$V$ is $\mathbf{H}$-Lipschitz continuous with constant $\eta \geq 1$,*
(vi) *$2(1 - 2\gamma + \delta^2)^{1/2} + 2\theta + \rho\beta + (1 - 2\rho\kappa + \rho^2\lambda^2\eta^2)^{1/2} < 1$.*

*Then there exist $x \in H$ and $y \in V(x)$ which are solutions of problem (1.1) and $x_n \longrightarrow x$, $y_n \longrightarrow y$ where sequences $\{x_n\}, \{y_n\}$ are generated by (4.1) and (4.2), respectively.*

PROOF. By inequality (4.1), (4.2), and Lemma 2.5. we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_n - Gx_n + m(x_n) + P_X(Gx_n - \rho(Tx_n + Ay_n) - m(x_n)) \\ &\quad - (x_{n-1} - Gx_{n-1} + m(x_{n-1})) \\ &\quad - P_X(Gx_{n-1} - \rho(Tx_{n-1} + Ay_{n-1}) - m(x_{n-1}))\| \\ &\le \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| \\ &\quad + \rho\|Tx_n - Tx_{n-1}\| + \|x_n - x_{n-1} - \rho(Ay_n - Ay_{n-1})\| \\ &= 2\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \rho\|Tx_n - Tx_{n-1}\| + \|x_n - x_{n-1} - \rho(Ay_n - Ay_{n-1})\|. \end{aligned} \tag{4.3}$$

By conditions (i)–(v), we have

$$\begin{aligned}\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\|^2 &= \|x_n - x_{n-1}\|^2 + \|Gx_n - Gx_{n-1}\|^2 \\ &\quad - 2(x_n - x_{n-1}, Gx_n - Gx_{n-1}) \\ &\le (1 - 2\gamma + \delta^2)\|x_n - x_{n-1}\|^2, \end{aligned} \tag{4.4}$$

$$\begin{aligned}\|x_n - x_{n-1} - \rho(Ay_n - Ay_{n-1})\|^2 &= \|x_n - x_{n-1}\|^2 + \rho^2\|Ay_n - Ay_{n-1}\|^2 \\ &\quad - 2\rho(x_n - x_{n-1}, Ay_n - Ay_{n-1}) \\ &\le \|x_n - x_{n-1}\|^2 + \lambda^2\rho^2\|y_n - y_{n-1}\|^2 \\ &\quad - 2\kappa\rho\|x_n - x_{n-1}\|^2 \\ &\le \|x_n - x_{n-1}\|^2 + \lambda^2\rho^2\mathbf{H}^2(V(x_n), V(x_{n-1})) \\ &\quad - 2\kappa\rho\|x_n - x_{n-1}\|^2 \\ &\le (1 - 2\kappa\rho + \lambda^2\eta^2\rho^2)\|x_n - x_{n-1}\|^2, \end{aligned} \tag{4.5}$$

$$\|m(x_n) - m(x_{n-1})\| \le \theta\|x_n - x_{n-1}\|, \tag{4.6}$$

$$\|Tx_n - Tx_{n-1}\| \le \beta\|x_n - x_{n-1}\|. \tag{4.7}$$

Now it follows from (4.3)–(4.7) that

$$\|x_{n+1} - x_n\| \le \tau\|x_n - x_{n-1}\|, \tag{4.8}$$

where $0 < \tau = 2(1 - 2\gamma + \delta^2)^{1/2} + 2\theta + \rho\beta + (1 - 2\rho\kappa + \rho^2\lambda^2\eta^2)^{1/2} < 1$ by (vi).

Consequently, $\{x_n\}$ is a Cauchy sequence in $H$. Let $x_n \longrightarrow x$. By (4.1), we have

$$\|y_n - y_{n+1}\| \le \mathbf{H}(V(x_n), V(x_{n+1})) \le \eta\|x_n - x_{n+1}\|,$$

and hence $\{y_n\}$ is also a Cauchy sequence in $H$. Let $\{y_n\} \longrightarrow y$. Since $P_X$, $G$, $T$, $A$, $V$, and $m$ are continuous in $H$, we have

$$x = x - Gx + m(x) + P_X(Gx - \rho(Tx + Ay) - m(x)).$$

It remains to show that $y \in V(x)$. In fact,

$$\begin{aligned} d(y, V(x)) &\le \|y - y_n\| + d(y_n, V(x)) \\ &\le \|y - y_n\| + H(V(x_n), V(x)) \\ &\le \|y - y_n\| + \eta\|x_n - x\|, \end{aligned} \tag{4.9}$$

where $d(y, V(x)) = \inf\{\|y - z\| : z \in V(x)\}$. By letting $n \longrightarrow \infty$, we have $d(y, V(x)) = 0$. Since $V(x) \in C(H)$, $y \in V(x)$. The result then follows from Theorem 3.3.

We note that since $\delta \geq \gamma$, $1-2\gamma+\delta^2 \geq 1-2\delta+\delta^2 = (1-\delta)^2 \geq 0$. Similarly, $1-2\kappa\rho+\lambda^2\eta^2\rho^2 \geq 1-2\lambda\eta\rho+\lambda^2\eta^2\rho^2 = (1-\lambda\eta\rho)^2 > 0$. Let $\xi = (1-2\gamma+\delta^2)^{1/2}+\theta$. If we suppose that $\xi < 1/2$, $\lambda\eta > (\beta^2+\frac{\kappa\beta}{1-2\xi})^{1/2}$, $\eta \geq 1$, $\kappa \geq \beta(1-2\xi)+2[\xi(1-\xi)(\lambda^2\eta^2-\beta^2)]^{1/2}$ and choose $\rho$ such that

$$a < \rho < b \tag{4.10}$$

where

$$a = \frac{\kappa-\beta(1-2\xi)-[(\kappa-\beta(1-2\xi)^2-4\xi(1-\xi)(\lambda^2\eta^2-\beta^2)]^{1/2}}{\lambda^2\eta^2-\beta^2},$$

$$b = \min\left\{\frac{1-2\xi}{\beta}, \frac{\kappa-\beta(1-2\xi)+[(\kappa-\beta(1-2\xi))^2-4\xi(1-\xi)(\lambda^2\eta^2-\beta^2)]^{1/2}}{\lambda^2\eta^2-\beta^2}\right\},$$

then condition (vi) of Theorem 4.1 will be satisfied. To see that there exists $\rho$ satisfying (4.10), we note that since $\tau < 1$, we must have

$$0 < \rho < \frac{1-2\xi}{\beta}.$$

On the other hand, since

$$(1-2\rho\kappa+\lambda^2\eta^2\rho^2)^{1/2} < 1-2\xi-\rho\beta,$$
$$(\lambda^2\eta^2)\rho^2-2(\kappa-\beta(1-2\xi))\rho+4\xi(1-\xi) < 0,$$

from which it follows

$$\left|\rho-\frac{\kappa-\beta(1-2\xi)}{\lambda^2\eta^2-\beta^2}\right| < \frac{[(\kappa-\beta(1-2\xi))^2-4\xi(1-\xi)(\lambda^2\eta^2-\beta^2)]^{1/2}}{\lambda^2\eta^2-\beta^2}.$$

Note that by assumption,

$$\frac{\kappa-\beta(1-2\xi)-[(\kappa-\beta(1-2\xi))^2-4\xi(1-\xi)(\lambda^2\eta^2-\beta^2)]^{1/2}}{\lambda^2\eta^2-\beta^2} < \frac{1-2\xi}{\beta}.$$

Hence, there exists positive $\rho$ satisfying (4.10).

The following result weakens condition of $A$ and strengthens condition of $T$ in Theorem 4.1.

THEOREM 4.2. *Let $H$ be a real Hilbert space, $X$ a nonempty closed convex subset of $H$, $T, A, G, m : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x) = m(x) + X$ for all $x \in H$. Suppose that the following conditions are satisfied:*

(i) *$T$ is both strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta$,*
(ii) *$G$ is both strongly monotone with constant $\gamma > 0$ and Lipschitz continuous with constant $\delta$,*
(iii) *$A$ is Lipschitz continuous with constant $\lambda$,*
(iv) *$m$ is Lipschitz continuous with constant $\theta$,*
(v) *$V$ is* **H**-*Lipschitz continuous with constant $\eta > 0$,*
(vi) $2(1-2\gamma+\delta^2)^{1/2}+2\theta+(1-2\rho\alpha+\rho^2\beta^2)^{1/2}+\rho\lambda\eta < 1$.

*Then there exist $x \in H$ and $y \in V(x)$ which are solutions of problem (1.1), and $x_n \longrightarrow x$, $y_n \longrightarrow y$ where sequences $\{x_n\}$, $\{y_n\}$ are generated by (4.1) and (4.2), respectively.*

PROOF. By inequality (4.1), (4.2), and Lemma 2.5, we have

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|x_n - Gx_n + m(x_n) + P_X(Gx_n - \rho(Tx_n + Ay_n) - m(x_n)) \\ &\quad - (x_{n-1} - Gx_{n-1} + m(x_{n-1})) \\ &\quad - P_X(Gx_{n-1} - \rho(Tx_{n-1} + Ay_{n-1}) - m(x_{n-1}))\| \\ &\leq \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + \|m(x_n) - m(x_{n-1})\| \\ &\quad + \|Gx_n - Gx_{n-1} - \rho(Tx_n - Tx_{n-1}) - \rho(Ay_n - Ay_{n-1}) \\ &\qquad - m(x_n) - m(x_{n-1})\| \\ &\leq 2\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \|x_n - x_{n-1} - \rho(Tx_n - Tx_{n-1})\| + \rho\|Ay_n - Ay_{n-1}\|. \end{aligned} \tag{4.11}$$

By conditions (i)–(v), we have

$$\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\|^2 \le (1 - 2\gamma + \delta^2)\|x_n - x_{n-1}\|^2, \tag{4.12}$$

$$\begin{aligned}\|x_n - x_{n-1} - \rho(Tx_n - Tx_{n-1})\|^2 &\le \|x_n - x_{n-1}\|^2 + \rho^2\|Tx_n - Tx_{n-1}\|^2 \\ &\quad - 2\rho(x_n - x_{n-1}, Tx_n - Tx_{n-1}) \\ &\le (1 - 2\rho\alpha + \rho^2\beta^2)\|x_n - x_{n-1}\|^2,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\|Ay_n - Ay_{n-1}\| &\le \lambda\|y_n - y_{n-1}\| \\ &\le \lambda \mathbf{H}(V(x_n), V(x_{n-1})) \\ &\le \lambda\eta\|x_n - x_{n-1}\|,\end{aligned} \tag{4.14}$$

$$\|m(x_n) - m(x_{n-1})\| \le \theta\|x_n - x_{n-1}\|. \tag{4.15}$$

Now it follows from (4.11)–(4.15) that

$$\|x_{n+1} - x_n\| \le \tau\|x_n - x_{n-1}\|,$$

where $0 < \tau = 2(1 - 2\gamma + \delta^2)^{1/2} + 2\theta + (1 - 2\rho\alpha + \rho^2\beta^2)^{1/2} + \rho\lambda\eta < 1$ by (vi).

Consequently, $\{x_n\}$ is a Cauchy sequence in $H$. The result then follows by employing the same argument as that of Theorem 4.1.

We note that since $\beta \ge \alpha$, $1 - 2\rho\alpha + \rho^2\beta^2 \ge (1 - \rho\beta)^2 \ge 0$. Let $\xi = (1 - 2\gamma + \delta^2)^{1/2} + \theta$.

If we suppose $\alpha \ge \lambda\eta(1 - 2\xi) + 2[\xi(1 - \xi)(\beta^2 - \lambda^2\eta^2)]^{1/2}$, $\xi < 1/2$, $\beta > (\lambda^2\eta^2 + \frac{\alpha\lambda\eta}{1-2\xi})^{1/2}$ and choose $\rho$ such that

$$c < \rho < d,$$

where

$$c = \frac{\alpha - \lambda\eta(1 - 2\xi) - [(\alpha - \lambda\eta(1 - 2\xi))^2 - 4\xi(1 - \xi)(\beta^2 - \lambda^2\eta^2)]^{1/2}}{\beta^2 - \lambda^2\eta^2},$$

$$d = \min\left\{\frac{1 - 2\xi}{\lambda\eta}, \frac{\alpha - \lambda\eta(1 - 2\xi) + [(\alpha - \lambda\eta(1 - 2\xi))^2 - 4\xi(1 - \xi)(\beta^2 - \lambda^2\eta^2)]^{1/2}}{\beta^2 - \lambda^2\eta^2}\right\},$$

then condition (vi) of Theorem 4.2 will be satisfied.

The next result weakens conditions of $T$ and $A$ in Theorem 4.1.

THEOREM 4.3. *Let $H$ be a real Hilbert space, $X$ a nonempty closed convex subset of $H$, $T, G, A, m : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x) = m(x) + X$ for all $x \in H$. Suppose that the following conditions are satisfied:*

(i) *$T$ is Lipschitz continuous with constant $\beta$,*
(ii) *$G$ is both strongly monotone with constant $\gamma > 0$ and Lipschitz continuous with constant $\delta$,*
(iii) *$A$ is Lipschitz continuous with constant $\lambda$,*
(iv) *$V$ is $\mathbf{H}$-Lipschitz continuous with constant $\eta > 0$,*
(v) *$m$ is Lipschitz continuous with constant $\theta$,*
(vi) $(1 - 2\gamma + \delta^2)^{1/2} + 2\theta + \delta + \rho(\beta + \lambda\eta) < 1$.

*Then there exist $x \in H$ and $y \in V(x)$ which are solutions of problem (1.1) and $x_n \longrightarrow x$, $y_n \longrightarrow y$ where sequences $\{x_n\}$, $\{y_n\}$ are generated by (4.1) and (4.2), respectively.*

PROOF. By inequality (4.1), (4.2), and Lemma 2.5, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_n - Gx_n + m(x_n) + P_X(Gx_n - \rho(Tx_n + Ay_n) - m(x_n)) \\ &\quad - (x_{n-1} - Gx_{n-1} + m(x_{n-1})) \\ &\quad - P_X(Gx_{n-1} - \rho(Tx_{n-1} + Ay_{n-1}) - m(x_{n-1}))\| \\ &\le \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \|Gx_n - Gx_{n-1}\| + \rho\|Tx_n - Tx_{n-1}\|\| + \rho\|Ay_n - Ay_{n-1}\|.\end{aligned} \tag{4.16}$$

By conditions (i)–(v), we have

$$\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\|^2 \le (1 - 2\gamma + \delta^2)\|x_n - x_{n-1}\|^2, \tag{4.17}$$

$$\|Tx_n - Tx_{n-1}\| \le \beta\|x_n - x_{n-1}\|, \tag{4.18}$$

$$\|Ay_n - Ay_{n-1}\| \le \lambda\eta\|x_n - x_{n-1}\|, \tag{4.19}$$

$$\|m(x_n) - m(x_{n-1})\| \le \theta\|x_n - x_{n-1}\|. \tag{4.20}$$

Now it follows from (4.16)–(4.20) that

$$\|x_{n+1} - x_n\| \le \tau\|x_n - x_{n-1}\|,$$

where $0 < \tau = (1 - 2\gamma + \delta^2)^{1/2} + 2\theta + \delta + \rho\beta + \rho\lambda\eta < 1$ by (vi). Hence, $\{x_n\}$ is a Cauchy sequence in $H$. The result then follows by employing the same argument as that of Theorem 4.1.

We note that if we suppose $\xi = (1 - 2\gamma + \delta^2)^{1/2} + 2\theta + \delta < 1$ and choose any $\rho$ such that $0 < \rho < \frac{1-\xi}{\beta+\lambda\eta}$, then condition (vi) of Theorem 4.3 will be satisfied.

## 5. GENERALIZED MULTIVALUED COMPLEMENTARITY PROBLEMS

Let $H$ be a real Hilbert space, $X$ be a closed convex cone of $H$, $X^*$ be the dual cone of $X$, $T, G, A : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x) = m(x) + X$ for all $x \in H$ where $m : H \to H$ is a point-to-point mapping. We shall study in this section the following nonlinear complementarity problems: find $x \in H$ and $y \in V(x)$ such that $Gx \in K(x)$ and

$$Tx + Ay \in K^*(x), \quad (Tx + Ay, Gx - m(x)) = 0. \tag{5.1}$$

Before we proceed any further, we make the following obervations.

(i) If $G$ is the identity mapping on $H$, then problem (5.1) is equivalent to finding $x \in K(x)$ and $y \in V(x)$ such that

$$Tx + Ay \in K^*(x), \quad (Tx + Ay, x - m(x)) = 0, \tag{5.2}$$

which are known as generalized strongly nonlinear quasi-complementarity problems studied by Chang and Huang [12].

(ii) If $G$ and $V$ are identity mappings on $H$, then problem (5.1) is equivalent to finding $x \in K(x)$ such that

$$Tx + Ax \in K^*(x), \quad (Tx + Ax, x - m(x)) = 0, \tag{5.3}$$

which are known as strongly nonlinear quasi-complementarity problems studied by Noor [13].

(iii) If $G$ and $V$ are identity mappings on $H$, $A \equiv 0$, then problem (5.1) is equivalent to finding $x \in K(x)$ such that

$$Tx \in K^*(x), \quad (Tx, x - m(x)) = 0, \tag{5.4}$$

which are known as generalized quasi-complementarity problems studied by Noor [14].

(iv) If $G$ and $V$ are identity mappings on $H$, $m \equiv 0$, then problem (5.1) is equivalent to finding $x \in X$ such that

$$Tx + Ax \in X^*, \quad (Tx + Ax, x) = 0, \tag{5.5}$$

which are known as mildly nonlinear complementarity problems studied by Noor [15].

(v) If $A \equiv 0$, $V$ is the identity mapping on $H$, $m \equiv 0$, then problem (5.1) is equivalent to finding $x \in H$ such that $Gx \in X$ and

$$Tx \in X^*, \quad (Tx, Gx) = 0, \tag{5.6}$$

which are known as general nonlinear complementarity problems studied by Noor [16,17].

(vi) If $G$ and $V$ are identity mappings on $H$, $A \equiv 0$, $m \equiv 0$, then problem (5.1) is equivalent to finding $x \in X$ and
$$Tx \in X^*, \quad (Tx, x) = 0, \tag{5.7}$$
which are known as generalized complementarity problems studied by Habetler and Price [18] and Karamardian [19].

(vii) If $T \equiv 0$, $m \equiv 0$, $G$ and $A$ are identity mappings on $H$, then problem (5.1) is equivalent to finding $x \in X$ and $y \in V(x)$ such that
$$y \in X^*, \quad (x, y) = 0, \tag{5.8}$$
which has been studied by Saigal [20] in finite-dimensional spaces.

In summary, problems (5.2)–(5.8) are special cases of problem (5.1). The following result states that problem (5.1) is equivalent to problem (1.1) with $K(x) = m(x) + X$ for all $x \in H$.

**Theorem** 5.1. *Let $H$ be a real Hilbert space, $X$ be a closed convex cone of $H$, $T, A, G, m : H \to H$, $V : H \to C(H)$, $K : H \to 2^H$ such that $K(x) = m(x) + X$ for all $x \in H$. Suppose $x \in H$ and $y \in V(x)$ are solutions of problem (1.1) if and only if $x \in H$, $y \in V(x)$ are solutions of problem (5.1).*

**Proof**. Suppose that $x \in H$ and $y \in V(x)$ solve problem (1.1). Then $Gx \in K(x)$ and
$$(Tx + Ay, z - Gx) \geq 0, \qquad \text{for all } z \in K(x). \tag{5.9}$$
Since $0 \in X$, $m(x) \in K(x)$, and hence
$$(Tx + Ay, m(x) - Gx) \geq 0. \tag{5.10}$$
To see $(Tx + Ay, m(x) - Gx) \leq 0$, we show that $z = 2Gx - m(x) \in K(x)$. Since $Gx \in K(x)$,
$$Gx - m(x) \in X.$$
Note that $X$ is a cone; we then have
$$2(Gx - m(x)) \in X.$$
Consequently,
$$2Gx - m(x) = m(x) + 2(Gx - m(x)) \in K(x),$$
and thus by substituting $2Gx - m(x)$ into (5.9), we then have
$$(Tx + Ay, m(x) - Gx) \leq 0. \tag{5.11}$$
Combining (5.10) and (5.11), we get
$$(Tx + Ay, Gx - m(x)) = 0.$$
For any $u \in X$, $z = m(x) + u \in K(x)$ and by (5.9), we have
$$\begin{aligned} 0 &\leq (Tx + Ay, z - Gx) \\ &= (Tx + Ay, m(x) + u - Gx) \\ &= (Tx + Ay, m(x) - Gx) + (Tx + Ay, u) \\ &= (Tx + Ay, u). \end{aligned}$$
Hence $Tx + Ay \in X^*$. On the other hand, taking $z = m(x) + G(x) \in K(x)$, we have
$$\begin{aligned} 0 &\leq (Tx + Ay, z - Gx) \\ &= (Tx + Ay, m(x) + Gx - Gx) \\ &= (Tx + Ay, m(x)). \end{aligned}$$

This implies that $Tx+Ay \in m^*(x)$. It follows from Lemma 2.7 that $Tx+Ay \in K^*(x)$. Therefore, $x$ and $y$ solve problem (5.1).

Conversely, suppose that $x \in H$ and $y \in V(x)$ solve (5.1). Since $Tx + Ay \in K^*(x)$ and $K(x) = m(x) + X$. For any $z \in K(x)$, $z = m(x) + u$ for some $u \in X$. We then have

$$\begin{aligned}(Tx + Ay, z - Gx) &= (Tx + Ay, m(x) + u - Gx)\\ &= (Tx + Ay, m(x) - Gx) + (Tx + Ay, u)\\ &= (Tx + Ay, u)\\ &\geq 0.\end{aligned}$$

Hence $x \in H$ and $y \in V(x)$ solve problem (1.1). The proof is completed.

By Theorem 5.1 and Theorems 4.1–4.3, we have the following existence and convergence results for problem (5.1).

THEOREM 5.2. *Under the hypotheses of Theorem 4.1, there exist $x \in H$, $y \in V(x)$ which solve problem (5.1) and $x_n \longrightarrow x$, $y_n \longrightarrow y$ where sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4.1) and (4.2), respectively.*

THEOREM 5.3. *Under the hypotheses of Theorem 4.2, there exist $x \in H$, $y \in V(x)$ which solve problem (5.1) and $x_n \longrightarrow x$, $y_n \longrightarrow y$ where sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4.1) and (4.2), respectively.*

THEOREM 5.4. *Under the hypotheses of Theorem 4.3, there exist $x \in H$, $y \in V(x)$ which solve problem (5.1) and $x_n \longrightarrow x$, $y_n \longrightarrow y$ where sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4.1) and (4.2), respectively.*

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